J. Math. Anal. Appl. 363 (2010) 697–710

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

$W_0^{1,p}$ versus C^1 local minimizers for a singular and critical functional

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ARTICLE INFO

Article history:
Received 13 March 2009
Available online 7 October 2009
Submitted by V. Radulescu

Keywords:
Quasilinear singular problems
Variational methods
Local minimizers
Strong comparison principle

ABSTRACT

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, $1 < p < +\infty$, $0 < \delta < 1$, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a C^1 function with $f(x,s) \geqslant 0$, $\forall (x,s) \in \Omega \times \mathbb{R}^+$ and $\sup_{x\in\Omega} f(x,s) \leqslant C(1+s)^q$, $\forall s \in \mathbb{R}^+$, for some $0 < q$ satisfying $q \leqslant p^* - 1$ if $p < N$. Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ continuous on $(0,+\infty)$ nonincreasing and satisfying $c_1 = \liminf_{t\to 0^+} g(t)t^\delta \leqslant \limsup_{t\to 0^+} g(t)t^\delta = c_2$, for some $c_1, c_2 > 0$. Consider the singular functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined as $I(u) \stackrel{\text{def}}{=} \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} - \int_\Omega F(x,u^+) - \int_\Omega G(u^+)$ where $F(x,u) = \int_0^s f(x,s)\,ds$, $G(u) = \int_0^s g(s)\,ds$. Theorem 1.1 proves that if $u_0 \in C^1(\overline{\Omega})$ satisfying $u_0 \geqslant \eta\, \mathrm{dist}(x,\partial\Omega)$, for some $0 < \eta$, is a local minimum of I in the $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ topology, then it is also a local minimum in $W_0^{1,p}(\Omega)$ topology. This result is useful for proving multiple solutions to the associated Euler–Lagrange equation (P) defined below. Theorem 1.1 generalises some results in Giacomoni, Schindler and Takáč (2007) [17] and due to the new proof given in the present paper can be also extended to more general quasilinear elliptic equations.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$ be a bounded smooth domain, $1 < p < +\infty$, $0 < \delta < 1$. Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a C^1 function satisfying:

(f1) $f(x,s) \geqslant 0$ for $(x,s) \in \overline{\Omega} \times \mathbb{R}^+$ and $f(x,0) = 0$.

(f2) There exists $q > p - 1$ satisfying $q \leqslant p^* - 1 \stackrel{\text{def}}{=} \frac{Np}{N-p} - 1$ if $p < N$, $q < \infty$ otherwise, such that $f(x,s) \leqslant C(1+s)^q$ for all $(x,s) \in \Omega \times \mathbb{R}^+$ and for some $C > 0$.

Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ continuous on $(0,+\infty)$ satisfying

(g1) g is nonincreasing on $(0,+\infty)$.
(g2) $c_1 \leqslant \liminf_{t\to 0^+} g(t)t^\delta \leqslant \limsup_{t\to 0^+} g(t)t^\delta = c_2$ for some $c_1, c_2 > 0$.

From (g2), g is singular at the origin and $\lim_{t\to 0^+} g(t) = +\infty$.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.10.012

Let $F(x,u) \stackrel{\text{def}}{=} \int_0^u f(x,s)\,\mathrm{d}s$ and $G(u) \stackrel{\text{def}}{=} \int_0^u g(s)\,\mathrm{d}s$. We consider the singular functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$I(u) \stackrel{\text{def}}{=} \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} - \int_\Omega F\big(x,u^+\big) - \int_\Omega G\big(u^+\big), \tag{1.1}$$

where as usual $t^+ \stackrel{\text{def}}{=} \max(t,0)$. Our aim in this paper is to show the following

Theorem 1.1. *Suppose that the conditions* (f1)–(f2) *and* (g1)–(g2) *are satisfied. Let $u_0 \in C^1(\overline{\Omega})$ satisfying*

$$u_0 \geqslant \eta d(x,\partial\Omega) \quad \text{for some } \eta > 0 \tag{1.2}$$

be a local minimizer of I in $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ topology; that is,

$$\exists \epsilon > 0 \quad \text{such that} \quad u \in C^1(\overline{\Omega}) \cap C_0(\overline{\Omega}), \quad \|u-u_0\|_{C^1(\overline{\Omega})} < \epsilon \quad \Rightarrow \quad I(u_0) \leqslant I(u).$$

Then, u_0 is a local minimizer of I in $W_0^{1,p}(\Omega)$ also.

From Lemma A.2 in Appendix A, we remark that the conditions on u_0 in the above theorem implies that u_0 satisfies in the distributions sense the Euler–Lagrange equation associated to I, that is

$$\text{(P)} \quad \begin{cases} -\Delta_p u = g(u) + f(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, & u > 0 \text{ in } \Omega. \end{cases}$$

It means that $u_0 \in W_0^{1,p}(\Omega)$ is a weak solution to (P), i.e. satisfies $\operatorname{ess\,inf}_K u_0 > 0$ over every compact set $K \subset \Omega$ and

$$\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\phi\,\mathrm{d}x = \int_\Omega g(u_0)\phi\,\mathrm{d}x + \int_\Omega f(x,u_0)\phi\,\mathrm{d}x \tag{1.3}$$

for all $\phi \in C_c^\infty(\Omega)$. As usual, $C_c^\infty(\Omega)$ denotes the space of all C^∞ functions $\phi : \Omega \to \mathbb{R}$ with compact support. We highlight that the condition (1.2) is natural. Indeed from Lemma A.5 in Appendix A, any weak solution to (P) satisfies (1.2) for some $\eta > 0$ independent of u. In particular, $u_0 \geqslant \underline{u}$ where $\underline{u}$ is the weak solution to the "pure" singular problem (PS):

$$\text{(PS)} \quad \begin{cases} -\Delta_p u = g(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, & u > 0 \text{ in } \Omega, \end{cases}$$

given by Lemma A.4. From Lemma A.4, $\underline{u}$ satisfies (1.2) and from Lemmas A.6, A.7 and B.1 $\underline{u} \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$. Using the approach introduced in Brezis and Nirenberg [4], used in Ambrosetti, Brezis and Cerami [1] and extended to the p-Laplacian case in Azorero, Manfredi and Peral [6] and in Guo and Zhang [19], Theorem 1.1 can be used to prove the existence of a second solution to (P) near u_0. Precisely, if f satisfies $\lim_{s\to+\infty} \frac{f(x,s)}{s^{p-1}} = +\infty$ uniformly in $x \in \overline{\Omega}$, Theorem 1.1 and assumptions (f1)–(f2), (g1)–(g2) prove that I has the Moutain Pass geometry (see Ambrosetti and Rabinowitz [2]) around u_0 and then admits a second critical point (consequently a second weak solution to (P)) as it is shown in Giacomoni, Schindler and Takac [17] for the particular case $g(s) = s^{-\delta}$, $f(x,s) = s^q$ with $0 < \delta < 1$, $p-1 < q < p^*-1$. To apply Theorem 1.1 in this context, we need to prove the existence of a C^1-minimizer of I. This follows from the strong comparison principle we state below in the singular case (see Theorem 2.3 in [17]):

Theorem 1.2. *Suppose that the conditions* (g1)–(g2) *are satisfied. Let $u, v \in C^{1,\beta}(\overline{\Omega})$, for some $0 < \beta < 1$, satisfy $0 \lneqq u$, $0 \lneqq v$ and*

$$-\Delta_p u - g(u) = f, \tag{1.4}$$

$$-\Delta_p v - g(v) = h, \tag{1.5}$$

with $u = v = 0$ on $\partial\Omega$, where $f, h \in C(\Omega)$ are such that $0 \leqslant f < h$ pointwise everywhere in Ω. Then, the following strong comparison principle holds:

$$0 < u < v \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial v}{\partial \nu} < \frac{\partial u}{\partial \nu} < 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

In respect to [17], we use different arguments to prove Theorem 1.1 which generalises Proposition 3.7 in [17].

Eq. (P) appears in several models: non-Newtonian flows in porous media, chemical heterogeneous catalysts, nonlinear heat equations (see Díaz, Morel and Oswald [10], Fulks and Maybee [13], Gamba and Jungel [14], Ghergu and Radulescu [15], see also Díaz [9], Leach and Needdham [22] and the overviews about singular elliptic equations: Hernández and Mancebo [20], Ghergu and Radulescu [16]) and then have intrinsic mathematical interest.

For proving Theorem 1.1, we will need uniform L^∞-estimates for a family of solutions to (P_ϵ) (see Section 2) as below.

Theorem 1.3. *Let $\{u_\epsilon\}_{\epsilon\in(0,1)}$ be a family of solutions to* (P_ϵ)*, where u_0 satisfies* (1.2) *and solves* (P)*; let $\sup_{\epsilon\in(0,1)}(\|u_\epsilon\|_{W_0^{1,p}(\Omega)}) < \infty$. Then, there exist $C_1, C_2 > 0$ (independent of ϵ) such that*

$$\sup_{\epsilon\in(0,1)} \|u_\epsilon\|_{L^\infty(\Omega)} < \infty \quad \textit{and} \quad C_1 d(x, \partial\Omega) \leqslant u_\epsilon \leqslant C_2 d(x, \partial\Omega).$$

The proof of the above theorem is a consequence of the results proved in Appendices A and B.

Concerning the case where $p = N$, we know from the Trudinger–Moser inequality that the critical growth is given by $e^{bu^{\frac{N}{N-1}}}$ for any $b > 0$. Then setting $f(x,u) = h(x,u)e^{bu^{\frac{N}{N-1}}}$ and assuming (g1)–(g2),

(h1) $h : \overline{\Omega} \times \mathbb{R}^+ \to [0,\infty)$ is a is C^1 and nonnegative with $h(x,0) = 0$,
(h2) $\liminf_{t\to\infty} h(x,t)e^{\epsilon|t|^{\frac{N}{N-1}}} = \infty$, $\liminf_{t\to\infty} h(x,t)e^{-\epsilon|t|^{\frac{N}{N-1}}} = 0$ uniformly in $x \in \overline{\Omega}$.

Theorem 1.1 holds. The proof is quite similar and uses in addition the Trudinger–Moser inequality in a similar way as Giacomoni, Prashanth and Sreenadh [18]. We don't give the proof of this case in the present paper.

Theorem 1.1 was proved first in [4] for the case of critical growth functionals I: $H_0^1(\Omega) \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, and later for critical growth functionals $I : W_0^{1,p}(\Omega) \to \mathbb{R}$, $1 < p < N$, $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$ in [6] and for critical and singular functionals in [17] in the special case $g(t) = t^{-\delta}$ and $f(x,t) = t^q$, with $\delta < 1$ and $1 < q \leqslant p^* - 1$. A key feature of these latter works is the uniform $C^{1,\alpha}$ estimate they obtain for equations like (P_ϵ) but involving two p-Laplace operators. We point out that *in this case* we cannot use in a straightforward way the classical result of Lieberman [23] where for a large class of degenerate elliptic equations (including p-Laplacian equations) the global $C^{1,\alpha}$ (up to the boundary) regularity is proved. Indeed, it is not clear if the operator in (P_ϵ) involving two p-Laplace operators satisfies both conditions (0.3a) and (0.3b) in Theorem 1 of Lieberman [23] (in case of only one operator, the both conditions are satisfied for $\kappa = 0$). In [6], the authors don't appeal the result of [23] (based on the use of Campanato spaces) but use the approach of freezing coefficients (see DiBenedetto [11]). Using constraints based on L^p-norms rather than Sobolev norms as in [6], the equations for which uniform estimates are required can be simplified to a standard type involving only one p-Laplace operator. This approach was followed in [7] and also adopted in this work to deal with the singular case (the previous works deal with the case where the nonlinearity is at least continuous). Moreover we slightly simplify and generalize the results in [17]. Since the quasilinear operator is not modified in (P_ϵ) in the proof of Theorem 1.1, Theorem 1.1 can be extended to more general quasilinear operators in divergence form and to nonisotropic operators (as the $p(x)$-Laplacian operator appearing in heterogeneous porous media models). This would provide existence of multiple solutions for such quasilinear singular elliptic equations.

2. $W^{1,p}$ versus C^1 local minimizers

Proof of Theorem 1.1. We first deal with the subcritical and then give the additional arguments to prove the result when $q = p^* - 1$. *Case 1*: $q < p^* - 1$. We use the arguments in [7].

Let $r \in (q, p^* - 1)$, define

$$K(w) = \frac{1}{r+1}\int_\Omega \left|w(x) - u_0(x)\right|^{r+1} \mathrm{d}x \quad \left(w \in W_0^{1,p}(\Omega)\right),$$

and

$$S_\epsilon \stackrel{\mathrm{def}}{=} \left\{v \in W_0^{1,p}(\Omega) \text{ such that } K(v) \leqslant \epsilon\right\}.$$

We consider the following constraint minimization problem:

$$I_\epsilon = \inf_{v\in S_\epsilon} I(v).$$

Clearly, we have that $I_\epsilon > -\infty$. Assume that the conclusion of Theorem 1.1 is not true. Then, from $q < p^* - 1$, the facts that I is weakly lower semicontinuous in $W_0^{1,p}(\Omega)$ and S_ϵ is closed and convex, it follows that for every $\epsilon \in (0,1)$ I_ϵ is achieved on some $v_\epsilon \in S_\epsilon$, that is $I(v_\epsilon) = I_\epsilon$ and $I(v_\epsilon) < I(u_0)$. Moreover, since $I(v_\epsilon^+) \leqslant I(v_\epsilon)$ and $v_\epsilon^+ \in S_\epsilon$, we may assume that $v_\epsilon \geqslant 0$.

We now consider the following two cases:

(1) Let $K(v_\epsilon) < \epsilon$. Then v_ϵ is also a local minimizer of I in $W_0^{1,p}(\Omega)$. We now show that I admits a Gâteaux-derivatives on v_ϵ to derive that v_ϵ satisfies the Euler–Lagrange equation associated with I. For this, according to Lemma A.2 in Appendix A, we need to prove that $\exists\tilde\eta > 0$ such that $v_\epsilon \geqslant \tilde\eta\,\mathrm{dist}(x, \partial\Omega)$ or equivalently

$$\exists\eta > 0 \quad \text{such that } v_\epsilon \geqslant \eta\varphi_1; \tag{2.1}$$

φ_1 is the normalized positive eigenfunction associated to

$$\lambda_1 = \lambda_1(\Omega) \stackrel{\text{def}}{=} \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega u^p}.$$

To prove (2.1), similarly to the proof of Lemma A.4 we argue by contradiction: $\forall \eta > 0$ let $\Omega_\eta = Supp\{(\eta\varphi_1 - v_\epsilon)^+\}$ and suppose that Ω_η has a nonzero measure.

Let $v_\eta = (\eta\varphi_1 - v_\epsilon)^+$ and for $0 < t \leqslant 1$ set $\xi(t) = I(v_\epsilon + tv_\eta)$. Then, there exists $c(t)$ satisfying $c(t) \geqslant \eta t$ such that $v_\epsilon + tv_\eta \geqslant c(t)\varphi_1$ for $t > 0$. Then, from Lemma A.2, ξ is differentiable for $0 < t \leqslant 1$ and $\xi'(t) = \langle I'(v_\epsilon + tv_\eta), v_\eta\rangle$. Thus,

$$\xi'(t) = \big\langle -\Delta_p(v_\epsilon + tv_\eta) - g(v_\epsilon + tv_\eta) - f(x, v_\epsilon + tv_\eta), v_\eta\big\rangle.$$

From (f1) and (g2), we see that

$$\xi'(1) = \big\langle I'(v_\epsilon + v_\eta), v_\eta\big\rangle = \big\langle I'(\eta\varphi_1), v_\eta\big\rangle = \big\langle -\Delta_p(\eta\varphi_1) - g(\eta\varphi_1) - f(x, \eta\varphi_1), v_\eta\big\rangle < 0$$

for $\eta > 0$ small enough.

Moreover,

$$\begin{aligned}&-\xi'(1) + \xi'(t)\\ &\quad = \big\langle -\Delta_p(v_\epsilon + tv_\eta) + \Delta_p(v_\epsilon + v_\eta)\big(g(v_\epsilon + tv_\eta) - g(v_\epsilon + v_\eta) - \big[f(x, v_\epsilon + tv_\eta) - f(x, v_\epsilon + v_\eta)\big]\big), v_\eta\big\rangle.\end{aligned}$$

Since $g(s) + f(x,s)$ is nonincreasing for $0 < s$ small enough uniformly to $x \in \Omega$ (by (f1), (g1)–(g2)) and from the monotonicity of $-\Delta_p$, we have that for $0 < \eta$ small enough $0 \leqslant \xi'(1) - \xi'(t)$. Moreover from Taylor's expansion and since $K(v_\epsilon) < \epsilon$, there exists $0 < \theta < 1$ such that

$$0 \leqslant I(v_\epsilon + v_\eta) - I(v_\epsilon) = \big\langle I'(v_\epsilon + \theta v_\eta), v_\eta\big\rangle = \xi'(\theta). \tag{2.2}$$

Letting $t = \theta$ we have $\xi'(\theta) \leqslant \xi'(1) < 0$. We obtain a contradiction with (2.2) and then $v_\epsilon \geqslant \eta\varphi_1$ for some $\eta > 0$ (which depends a priori on ϵ). Since v_ϵ is a local minimizer of I, and I is Gâteaux differentiable in v_ϵ, we get $I'(v_\epsilon)$ is defined and $I'(v_\epsilon) = 0$. Recalling that $\underline{u}$ is the solution to (PS) given by Lemma A.4 and from the weak comparison principle, we have that $\eta\varphi_1 \leqslant \underline{u} \leqslant v_\epsilon$ for some $\eta > 0$ (independent of ϵ). Since $v_\epsilon \in S_\epsilon$ and from the fact that v_ϵ satisfies (P), we get that $\{v_\epsilon\}_{\epsilon \geqslant 0}$ is uniformly bounded in $W_0^{1,p}(\Omega)$. Now, using Lemmas A.6, A.7 and Theorem B.1 in Appendices A and B, we get

$$|v_\epsilon|_{C^{1,\alpha}(\overline{\Omega})} \leqslant C \tag{2.3}$$

and as $\epsilon \to 0^+$,

$$v_\epsilon \to u_0 \quad \text{in } C^1(\overline{\Omega})$$

which contradicts the fact that u_0 is a local minimizer in $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$.

Now, we deal with the *second case*:

(2) $K(v_\epsilon) = \epsilon$.

We again show that $v_\epsilon \geqslant \eta\varphi_1$ in Ω for some $\eta > 0$. Taking $v_\eta = (\eta\varphi_1 - v_\epsilon)^+$, $\xi(t) = I(v_\epsilon + tv_\eta)$, we obtain as above that $\xi'(t) \leqslant \xi'(1) < 0$ for $0 < t < 1$ and $0 < \eta$ small enough.

Then $\xi(t) = I(v_\epsilon + tv_\eta)$ is decreasing. This implies that $I(v_\epsilon) > I(v_\epsilon + tv_\eta)$ for $t > 0$ and using (1.2),

$$K(v_\epsilon + tv_\eta) = \frac{1}{r+1}\int_\Omega \big|u_0 - (v_\epsilon + tv_\eta)\big|^{r+1}\,dx \leqslant \int_\Omega |u_0 - v_\epsilon|^{r+1}\,dx = \epsilon.$$

This yields a contradiction with the fact that v_ϵ is a global minimizer of I on S_ϵ.

In this case, from the Lagrange multiplier rule we have

$$I'(v_\epsilon) = \mu_\epsilon K'(v_\epsilon), \quad \text{for some } \mu_\epsilon \in \mathbb{R}.$$

First assume that $\mu_\epsilon > 0$, then there exists $\varphi \in W_0^{1,p}(\Omega)$ such that

$$\big\langle I'(v_\epsilon), \varphi\big\rangle < 0 \quad \text{and} \quad \big\langle K'(v_\epsilon), \varphi\big\rangle < 0$$

and then for t small we have

$$\begin{cases} I(v_\epsilon + t\varphi) < I(v_\epsilon), \\ K(v_\epsilon + t\varphi) < K(v_\epsilon) = \epsilon. \end{cases} \tag{2.4}$$

This contradicts the fact that v_ϵ is a global minimizer of I in S_ϵ. It follows that $\mu_\epsilon \leqslant 0$.

We deal now with the following two cases:

Case (i): $\inf_{\epsilon\in(0,1)} \mu_\epsilon \stackrel{\text{def}}{=} \ell > -\infty$. In this case, using

$$(\mathrm{P}_\epsilon) - \Delta_p v_\epsilon = g(v_\epsilon) + f(x, v_\epsilon) + \mu_\epsilon\big(|v_\epsilon - u_0|^{r-1}(v_\epsilon - u_0)\big)$$

we conclude from the weak comparison principle to show that $\eta\varphi_1 \leqslant v_\epsilon$ with some $\eta > 0$ small enough, independent of ϵ (note that for $0 < \eta$ small enough and for all $\ell \leqslant \mu_\epsilon \leqslant 0$, we have that $\eta\varphi_1$ is a strict subsolution to (P_ϵ)).

Now, since $\ell \leqslant \mu_\epsilon \leqslant 0$, there exist $M, c > 0$ independent of ϵ such that

$$-\Delta_p(v_\epsilon - 1)^+ \leqslant M + c\big((v_\epsilon - 1)^+\big)^r.$$

Using the Moser iterations technique as in Lemma A.6 of Appendix A, we get that $|v_\epsilon|_{L^\infty} \leqslant C$ for some C independent of ϵ.

Using Lemma A.7 in Appendix A, we deduce that $v_\epsilon \leqslant k\varphi_1$ for some $k > 0$ independent of ϵ. From the uniform estimate $\eta\varphi_1 \leqslant v_\epsilon \leqslant k\varphi_1$, we can apply Theorem B.1 in Appendix B and get $|v_\epsilon|_{C^{1,\alpha}(\overline{\Omega})} \leqslant C$ for some constant $C > 0$ independent of ϵ. Then we conclude as above.

Let us consider *the case (ii)*: $\inf_{\epsilon\in(0,1)} \mu_\epsilon = -\infty$. From above, we can assume that $\mu_\epsilon \leqslant -1$ for $0 < \epsilon$ small enough. As above, we have that $v_\epsilon \geqslant \eta\varphi_1$ for $\eta > 0$ small enough and independent of ϵ. Furthermore, there exists a number $M > 0$, independent of ϵ, such that for

$$\gamma(s, x, t) \stackrel{\text{def}}{=} g(t) + f(x, t) + s\big|t - u_0(x)\big|^{r-1}\big(t - u_0(x)\big) \tag{2.5}$$

we have

$$\gamma(s, x, t) < 0, \quad \forall(s, x, t) \in (-\infty, -1] \times \Omega \times (M, +\infty).$$

Then, from the weak comparison principle we have that $v_\epsilon \leqslant M$ for $\epsilon > 0$ small enough. From Lemma A.2, since $u_0 \in W_0^{1,p}(\Omega)$ satisfies (1.2) and is a C^1 local minimizer, u_0 is a weak solution to (P) (defined in the Introduction), i.e. $\operatorname{ess\,inf}_K u_0 > 0$ over every compact set $K \subset \Omega$ and

$$\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\phi \,\mathrm{d}x = \int_\Omega g(u_0)\phi \,\mathrm{d}x + \int_\Omega f(x, u_0)\phi \,\mathrm{d}x$$

for all $\phi \in C_c^\infty(\Omega)$. From Lemma A.5, for every function $w \in W_0^{1,p}(\Omega)$, u_0 satisfies

$$\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla w \,\mathrm{d}x = \int_\Omega g(u_0)w \,\mathrm{d}x + \int_\Omega f(x, u_0)w \,\mathrm{d}x.$$

Similarly,

$$\int_\Omega |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon \cdot \nabla w \,\mathrm{d}x = \int_\Omega \gamma(\mu_\epsilon, x, v_\epsilon)w \,\mathrm{d}x.$$

Now, substracting the above relations with $w = (v_\epsilon - u_0)|v_\epsilon - u_0|^{\beta-1}$, where $\beta \geqslant 1$, we obtain

$$\begin{aligned}
0 &\leqslant \beta \int_\Omega \big(|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon - |\nabla u_0|^{p-2}\nabla u_0\big).\nabla(v_\epsilon - u_0)|v_\epsilon - u_0|^{\beta-1} \,\mathrm{d}x \\
&\leqslant \beta \int_\Omega \big(|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon - |\nabla u_0|^{p-2}\nabla u\big).\nabla(v_\epsilon - u_0)|v_\epsilon - u_0|^{\beta-1} \,\mathrm{d}x \\
&\quad - \int_\Omega \big(g(v_\epsilon) - g(u_0)\big)(v_\epsilon - u_0)|v_\epsilon - u_0|^{\beta-1} \,\mathrm{d}x \\
&= \int_\Omega \big(f(x, v_\epsilon) - f(x, u_0)\big)(v_\epsilon - u_0)|v_\epsilon - u_0|^{\beta-1} + \mu_\epsilon \int_\Omega |v_\epsilon - u_0|^{\beta+r} \,\mathrm{d}x.
\end{aligned}$$

Using the bounds about v_ϵ, u_0 and the Hölder inequality we get

$$-\mu_\epsilon \|v_\epsilon - u_0\|^r_{L^{\beta+r}(\Omega)} \leqslant C|\Omega|^{\frac{r}{r+\beta}},$$

where C does not depend on β and ϵ. Passing to the limit $\beta \to +\infty$ this leads to

$$-\mu_\epsilon \|v_\epsilon - u_0\|^r_{L^\infty(\Omega)} \leqslant C.$$

So the right-hand side of (2.5) is uniformly bounded in $L^\infty(\Omega)$ from which as in the first case, we obtain that v_ϵ is bounded in $C^{1,\alpha}(\overline{\Omega})$ independently of ϵ and we conclude as above. The proof of Theorem 1.1 in the subcritical case is now complete.

Now, we deal with the *critical case*, i.e. $q = p^* - 1$. Again, we assume that the conclusion of Theorem 1.1 is not true. As in [4], we first make a truncation argument to get the weak lower semicontinuity property of the energy functional. Precisely, assume by contradiction that u_0 is not a local minimizer of I in the $W_0^{1,p}(\Omega)$ topology. Let

$$\chi(w) = \frac{1}{p^*}\int_\Omega \left|w(x) - u_0\right|^{p^*} \mathrm{d}x \quad \left(w \in W_0^{1,p}(\Omega)\right),$$

and

$$\mathcal{C}_\varepsilon \stackrel{\text{def}}{=} \left\{v \in W_0^{1,p}(\Omega) \text{ such that } \chi(v) \leqslant \epsilon\right\}.$$

We now consider the truncated functional

$$I_j(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x - \int_\Omega G(v)\,\mathrm{d}x - \int_\Omega F_j(x, v)\,\mathrm{d}x, \quad \forall v \in W_0^{1,p}(\Omega) \tag{2.6}$$

for $j = 1, 2, \ldots,$ where $f_j(x, s) := f(x, T_j(s))$, $F_j(x, s) = \int_0^s f_j(x, t)\,\mathrm{d}t$ and

$$T_j(s) = \begin{cases} -j & \text{if } s \leqslant -j, \\ s & \text{if } -j \leqslant s \leqslant j, \\ +j & \text{if } s \geqslant j. \end{cases} \tag{2.7}$$

By the Lebesgue theorem, we have that for any $v \in W_0^{1,p}(\Omega)$,

$$I_j(v) \to I(v) \quad \text{as } j \to \infty.$$

It follows that for each $\epsilon > 0$, there is some j_ϵ (with $j_\epsilon \to +\infty$ as $\epsilon \to 0^+$) such that $I_{j_\varepsilon}(v_\epsilon) < I(u_0)$. On the other hand, since S_ϵ is closed and convex and from the fact that I_{j_ϵ} is weakly lower semicontinuous we deduce that I_{j_ϵ} achieves its infimum at some $u_\epsilon \in \mathcal{C}_\epsilon$. Therefore, for $0 < \epsilon$ small enough,

$$I_{j_\epsilon}(u_\epsilon) \leqslant I_{j_\epsilon}(v_\epsilon) < I_{j_\epsilon}(u_0) = I(u_0). \tag{2.8}$$

As above, we have that $\exists \eta > 0$ independent of ϵ such that

$$u_\epsilon \geqslant \eta\varphi_1. \tag{2.9}$$

From (2.9), I_{j_ϵ} admits a Gâteaux-derivative in u_ϵ and since u_ϵ a local minimizer of I_{j_ϵ}, we get that $I'_{j_\epsilon}(u_\epsilon)$ is defined and from the Lagrange multiplier rule, there exists $\mu_\epsilon \in \mathbb{R}^-$ such that $I'_{j_\epsilon}(u_\epsilon) = \mu_\epsilon \chi'(v_\epsilon)$.

By construction $u_\epsilon \to u_0$ in $L^{p^*}(\Omega)$ as $\epsilon \to 0$, and it follows from above that u_ϵ remains bounded in $W_0^{1,p}(\Omega)$.

Claim. *$\{u_\epsilon\}$ are uniformly bounded in $L^\infty(\Omega)$ as $\epsilon \to 0$.*

Assuming this claim, we can argue as above to derive a uniform bound of $\{u_\epsilon\}$ in $C^{1,\alpha}(\overline{\Omega})$. Then, it follows that for $\epsilon > 0$ sufficiently small,

$$I(u_\epsilon) = I_{j_\epsilon}(u_\epsilon) < I(u_0), \tag{2.10}$$

which contradicts the fact that u_0 is a local minimizer of I for the $C_0^1(\overline{\Omega})$ topology, and the proof of the Theorem 1.1 is complete.

Finally, let us prove Claim. For that, we again distinguish between the following two cases: *case (i)* $\inf_{\epsilon\in(0,1)} \mu_\epsilon > -\infty$; *case (ii)* $\inf_{\epsilon\in(0,1)} \mu_\epsilon = -\infty$.

In case (i) from the Euler equation (P_ϵ):

$$(P_\epsilon)\ -\Delta_p u = g(u) + f_{j_\epsilon}(x, u) + \mu_\epsilon\left(|u - u_0|^{p^*-2}(u - u_0)\right) \tag{2.11}$$

satisfied by u_ϵ we get that (see the first part of the proof of Lemma A.6 in Appendix A)

$$-\Delta_p(u_\epsilon - 1)^+ \leqslant M + c\left|(u_\epsilon - 1)^+\right|^{p^*-2}(u_\epsilon - 1)^+$$

for some $M > 0$ independent of ϵ. Now using the Moser iterations (observe that the singular term involving g is monotone and then the proof works similarly) as in García Azorero and Peral [5, pp. 950–953] (see also Lemma 3.7, step 1 in De Figueiredo, Gossez and Ubilla [8]), we get that $\{u_\epsilon\}$ are bounded in $L^{\beta p^*}(\Omega)$ for some $\beta > 1$ independently of ϵ. Then, using Theorem 7.1 in Ladyzenskaja and Ural'ceva [21, p. 263], we obtain that $\{u_\epsilon\}$ is uniformly bounded in $L^\infty(\Omega)$. This proves Claim in the case (i).

Let us consider now the case (ii). Again as in the subcritical case, we have that $u_\epsilon \geqslant \eta\varphi_1$ for some $\eta > 0$ independent of ϵ. Moreover, there exists a number $M > 0$, independent of ϵ, such that for

$$g(s) + f_{j_\epsilon}(x, s) + \mu_\epsilon \left|s - u_0(x)\right|^{p^*-2}\big(s - u_0(x)\big) < 0 \quad \text{if } s > M. \tag{2.12}$$

Taking now $(u_\epsilon - M)^+$ as a testing function in (2.11), one concludes by the weak comparison principle that $u_\epsilon(x) \leqslant M$ in Ω.

Then, the proof of the claim follows from the same arguments as in the subcritical case. This concludes the proof of Claim in case (ii). □

The results in Appendices A and B are some adaptations of results proved in [17] and used in the present paper for proving $L^\infty(\Omega)$ and $C^{1,\alpha}(\overline{\Omega})$ estimates. In particular Theorem 1.3 is a consequence of Lemmas A.6, A.7 and Theorem B.1.

Appendix A

We start with an important technical tool which enables us to estimate the singularity in the Gâteaux derivative of the energy functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined in (1.1).

Lemma A.1. *Let $0 < \delta < 1$. Then there exists a constant $C_\delta > 0$ such that the inequality*

$$\int_0^1 |\mathbf{a} + s\mathbf{b}|^{-\delta}\, \mathrm{d}s \leqslant C_\delta \Big(\max_{0 \leqslant s \leqslant 1} |\mathbf{a} + s\mathbf{b}| \Big)^{-\delta} \tag{A.1}$$

holds true for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ with $|\mathbf{a}| + |\mathbf{b}| > 0$.

An elementary proof of this lemma can be found in Takáč [25, Lemma A.1, p. 233].

We continue by showing the Gâteaux-differentiability of the energy functional I at a point $u \in W_0^{1,p}(\Omega)$ satisfying $u \geqslant \varepsilon\varphi_1$ in Ω with a constant $\varepsilon > 0$.

Lemma A.2. *Let the assumptions* (f1)–(f2) *and* (g1)–(g2) *be satisfied. Assume that $u, v \in W_0^{1,p}(\Omega)$ and u satisfies $u \geqslant \varepsilon\varphi_1$ in Ω with a constant $\varepsilon > 0$. Then we have $g(u)v \in L^1(\Omega)$ and*

$$\lim_{t\to 0} \frac{1}{t}\big(I(u + tv) - I(u)\big) = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, \mathrm{d}x - \int_\Omega g(u)v\, \mathrm{d}x - \int_\Omega f(x, u)v\, \mathrm{d}x. \tag{A.2}$$

Proof. We show the result only for the singular term $\int_\Omega g(u)v\, \mathrm{d}x$; the other two terms are treated in a standard way. So let

$$H(u) = \int_\Omega G\big(u(x)^+\big)\, \mathrm{d}x \quad \text{for } u \in W_0^{1,p}(\Omega).$$

For $\xi \in \mathbb{R} \setminus \{0\}$ we define

$$z(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi} G\big(\xi^+\big) = \begin{cases} g(\xi) & \text{if } \xi > 0, \\ 0 & \text{if } \xi < 0. \end{cases}$$

Consequently,

$$\frac{1}{t}\big(H(u + tv) - H(u)\big) = \int_\Omega \left(\int_0^1 z(u + stv)\, \mathrm{d}s \right) v\, \mathrm{d}x. \tag{A.3}$$

Notice that for almost every $x \in \Omega$ we have $u(x) > 0$ and

$$\int_0^1 z\big(u(x) + stv(x)\big)\, \mathrm{d}s \to z\big(u(x)\big) = g\big(u(x)\big) \quad \text{as } t \to 0.$$

Moreover, the integral on the left-hand side (with nonnegative integrand) is dominated by

$$\begin{aligned}\int_0^1 z\big(u(x)+stv(x)\big)\,ds &\leqslant C\int_0^1 \big|u(x)+stv(x)\big|^{-\delta}\,ds \\ &\leqslant C_\delta\Big(\max_{0\leqslant s\leqslant 1}\big|u(x)+stv(x)\big|\Big)^{-\delta} \\ &\leqslant C_\delta u(x)^{-\delta}\leqslant C_\delta\big(\varepsilon\varphi_1(x)\big)^{-\delta} \\ &= C_{\delta,\varepsilon}\varphi_1(x)^{-\delta}\end{aligned}$$

with constants $C, C_{\delta,\varepsilon}>0$ independent of $x\in\Omega$. Here, we have used the estimate (A.1) from Lemma A.1 above. Finally, we have $v\varphi_1^{-\delta}\in L^1(\Omega)$, by $v\in W_0^{1,p}(\Omega)$ and Hardy's inequality. Hence, we are allowed to invoke the Lebesgue dominated convergence theorem in (A.3) from which the lemma follows by letting $t\to 0$. $\square$

Corollary A.1. *Let the assumptions* (f1)–(f2) *and* (g1)–(g2) *be satisfied. Then the energy functional $I : W_0^{1,p}(\Omega)\to\mathbb{R}$ is Gâteaux-differentiable at every point $u\in W_0^{1,p}(\Omega)$ that satisfies $u\geqslant\varepsilon\varphi_1$ in Ω with a constant $\varepsilon>0$. Its Gâteaux derivative $I'(u)$ at u is given by*

$$\big\langle I'(u),v\big\rangle=\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx-\int_\Omega g(u)v\,dx-\int_\Omega f(x,u)v\,dx \tag{A.4}$$

for $v\in W_0^{1,p}(\Omega)$.

We continue by proving the C^1-differentiability of the cut off energy functional $\bar I$ defined below:

Lemma A.3. *Let the assumptions* (f1)–(f2) *and* (g1)–(g2) *be satisfied, and $w\in W_0^{1,p}(\Omega)$ such that $w\geqslant\epsilon\varphi_1$ with some $\epsilon>0$. Setting for $x\in\Omega$,*

$$h(x,s)=\begin{cases} g(s)+f(x,s) & \text{if } s\geqslant w(x),\\ g(w(x))+f(x,w(x)) & \text{if } s<w(x),\end{cases}$$

$H(x,s)=\int_0^s h(x,t)\,dt$ and for $u\in W_0^{1,p}(\Omega)$,

$$\bar I(u)=\frac{1}{p}\int_\Omega |\nabla u|^p\,dx-\int_\Omega H\big(x,u^+\big)\,dx,$$

we have that $\bar I$ belongs to $C^1(W_0^{1,p}(\Omega),\mathbb{R})$.

Proof. As in Lemma A.2, we concentrate on the singular term, the others being standard. Let

$$h(x,s)=\begin{cases} g(s) & \text{if } s\geqslant w(x),\\ g(w(x)) & \text{if } s<w(x),\end{cases}$$

$H(x,s)=\int_0^s h(x,t)\,dt$, and $S(u)=\int_\Omega H(x,u^+)\,dx$. Proceeding as in Lemma A.2, we obtain that for all $u\in W_0^{1,p}(\Omega)$, $S(u)$ has a Gâteaux derivative $S'(u)$ given by

$$\big\langle S'(u),v\big\rangle=\int_\Omega g\Big(\max\big\{u(x),w(x)\big\}\Big)v(x)\,dx.$$

Let $u_k\in W_0^{1,p}(\Omega)$, $u_k\to u_0$. Then

$$\begin{aligned}\big|\big\langle S'(u_k)-S'(u_0),v\big\rangle\big| &= \left|\int_\Omega \big(g\big(\max\big\{u_k(x),w(x)\big\}\big)v(x)-g\big(\max\big\{u(x),w(x)\big\}\big)v(x)\big)\,dx\right| \\ &\leqslant 2C\int_\Omega w^{-\delta}|v|\,dx \\ &\leqslant 2C\epsilon^{-\delta}\int_\Omega \varphi_1^{-\delta}|v|\,dx\end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$. Again, as in Lemma A.2, we use Hardy's inequality to deduce that $\varphi_1^{-\delta} v \in L^1(\Omega)$, so that by Lebesgue's dominated convergence theorem we conclude that the Gâteaux derivative of S is continuous which implies that $S \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$. $\square$

We give now the existence of a subsolution to (P):

Lemma A.4. *Assume assumptions* (g1)–(g2). *Then problem* (PS) *possesses a weak solution in $W_0^{1,p}(\Omega)$ in the sense of distributions. This solution, denoted by $\underline{u}$, is the unique global minimizer to the energy functional $\tilde{E}$ given by*

$$\tilde{E}(u) \stackrel{\text{def}}{=} \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \int_\Omega G(u^+) \, dx, \quad u \in W_0^{1,p}(\Omega)$$

in $W_0^{1,p}(\Omega)$ and $\underline{u} \geqslant \epsilon_0 \varphi_1$ a.e. in Ω, where $\epsilon_0 > 0$. In addition, $\underline{u}$ is the unique solution to (PS) *in $\Psi \stackrel{\text{def}}{=} \{u \in W_0^{1,p}(\Omega)$ such that $u \geqslant \eta\varphi_1$ for some $\eta > 0\}$.*

Proof. First, owing to the Poincaré inequality, assumption (g2) and $0 < 1-\delta < 1 < p < \infty$, the functional $\tilde{E}$ is coercive and weakly lower semicontinuous on $W_0^{1,p}(\Omega)$. It follows that $\tilde{E}$ possesses a global minimizer $\tilde{u} \in W_0^{1,p}(\Omega)$. We have $\tilde{u} \not\equiv 0$ in Ω, owing to $\tilde{E}(0) = 0 > \tilde{E}(\epsilon\varphi_1)$ for every $\epsilon > 0$ small enough.

Second, the polar decomposition $u = u^+ - u^-$ of any function $u \in W_0^{1,p}(\Omega)$ gives $\nabla u = \nabla u^+ - \nabla u^-$. Thus, if $\tilde{u}$ is a global minimizer for $\tilde{E}$, then so is its absolute value $|\tilde{u}|$, by $\tilde{E}(|\tilde{u}|) \leqslant \tilde{E}(\tilde{u})$. The equality $\tilde{E}(|\tilde{u}|) = \tilde{E}(\tilde{u})$ holds if and only if $\tilde{u}^- = 0$ a.e. in Ω, that is, if and only if $\tilde{u} \geqslant 0$ a.e. in Ω. Thus, any global minimizer $\tilde{u}$ for $\tilde{E}$ must satisfy $\tilde{u} \geqslant 0$ a.e. in Ω. Equivalently, $\tilde{u} \in W_0^{1,p}(\Omega)_+$ where

$$W_0^{1,p}(\Omega)_+ \stackrel{\text{def}}{=} \{u \in W_0^{1,p}(\Omega)\colon\ u \geqslant 0 \text{ a.e. in } \Omega\}$$

stands for the positive cone in $W_0^{1,p}(\Omega)$.

Third, we will show that even $\tilde{u} \geqslant \epsilon_0\varphi_1$ holds almost everywhere in Ω with a constant $\epsilon_0 > 0$ small enough. To this end, let us first remark that from Lemma A.2 the Gâteaux derivative $\tilde{E}'(\epsilon_0\varphi_1)$ of $\tilde{E}$ at $\epsilon\varphi_1$ exists and satisfies

$$\begin{aligned} \tilde{E}'(\epsilon_0\varphi_1) &= -\Delta_p(\epsilon_0\varphi_1) - g(\epsilon_0\varphi_1) \\ &= \lambda_1(\epsilon_0\varphi_1)^{p-1} - g(\epsilon_0\varphi_1) \\ &\leqslant (\epsilon_0\varphi_1)^{-\delta}\big(\lambda_1(\epsilon_0\varphi_1)^{p-1+\delta} - c_1\big) \\ &\leqslant -\frac{c_1}{2}(\epsilon_0\varphi_1)^{-\delta} < 0 \end{aligned} \tag{A.5}$$

whenever $\epsilon_0 > 0$ is small enough.

On the contrary to our claim above, suppose that the (nonnegative) function $v = (\tilde{u} - \epsilon_0\varphi_1)^- = (\epsilon_0\varphi_1 - \tilde{u})^+$ does not vanish identically in Ω. Denote

$$\Omega^+ = \{x \in \Omega\colon\ v(x) > 0\}.$$

Let us investigate the function $\xi(t) \stackrel{\text{def}}{=} \tilde{E}(\tilde{u} + tv)$ of $t \in \mathbb{R}_+ = [0, \infty)$. From assumption (g1) this function is convex thanks to the fact that the restriction of the functional $\tilde{E}$ to the positive cone $W_0^{1,p}(\Omega)_+$ is convex. We have $\xi(t) \geqslant \xi(0)$ for all $t \geqslant 0$. Furthermore, owing to $\tilde{u} + tv \geqslant \max\{\tilde{u}, t\epsilon_0\varphi_1\} \geqslant t\epsilon_0\varphi_1$ for $t > 0$, the Gâteaux derivative $\tilde{E}'(\tilde{u} + tv)$ of $\tilde{E}$ at $\tilde{u} + tv$ exists and yields $\xi'(t) = \langle \tilde{E}'(\tilde{u} + tv), v\rangle$ for $t > 0$. This derivative is nonnegative and nondecreasing. Consequently, for $0 < t < 1$ we have

$$0 \leqslant \xi'(1) - \xi'(t) = \big\langle \tilde{E}'(\tilde{u} + v) - \tilde{E}'(\tilde{u} + tv), v\big\rangle = \int_{\Omega^+} \tilde{E}'(\epsilon_0\varphi_1) v \, dx - \xi'(t) \leqslant -\frac{c_1}{2} \int_{\Omega^+} (\epsilon_0\varphi_1)^{-\delta} v \, dx < 0, \tag{A.6}$$

by inequality (A.5) and $\xi'(t) \geqslant 0$, a contradiction. We have verified $v \equiv 0$ in Ω, that is, $\tilde{u} \geqslant \epsilon_0\varphi_1$ a.e. in Ω.

Finally, we have proved that every global minimizer $\tilde{u}$ for $\tilde{E}$ on $W_0^{1,p}(\Omega)$ must satisfy $\tilde{u} \geqslant \epsilon_0\varphi_1$ a.e. in Ω.

From the fact $-\Delta_p(\cdot) - g(\cdot)$ is a monotone operator in the cone $\Psi \stackrel{\text{def}}{=} \{u \in W_0^{1,p}(\Omega)$ such that $u \geqslant \eta\varphi_1$ for some $\eta > 0\}$ and the weak comparison principle, we conclude that $\tilde{E}$ has a unique global minimizer denoted by $\underline{u}$ in $W_0^{1,p}(\Omega)$ with the property $\operatorname{ess\,inf}_K \underline{u} > 0$ for any compact set $K \subset \Omega$. $\underline{u}$ is then the unique weak solution to (PS) in Ψ and satisfies (1.2). $\square$

Next, we give some regularity results for weak solutions to problem (P). We start with the following lemma which allows for test functions ϕ in (P) to be taken in $W_0^{1,p}(\Omega)$ rather than only in $C_c^\infty(\Omega)$ $(\subset W_0^{1,p}(\Omega))$.

Lemma A.5. *Let the assumptions* (f1)–(f2) *and* (g1)–(g2) *be satisfied. Each positive weak solution* u *of problem* (P) *satisfies* $u \geqslant \epsilon\varphi_1$ *a.e. in* Ω*, where* $\epsilon > 0$ *is a constant independent of* u*. Moreover, for every function* $w \in W_0^{1,p}(\Omega)$ *we have* $g(u)w \in L^1(\Omega)$ *and*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w \, dx = \int_\Omega g(u)w \, dx + \int_\Omega f(x,u)w \, dx. \tag{A.7}$$

Proof. Let u be a positive weak solution of (P). Recall that u is required to satisfy $\operatorname{ess\,inf}_K u > 0$ over every compact set $K \subset \Omega$.

First, we establish the inequality

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w \, dx \geqslant \int_\Omega g(u)w \, dx + \int_\Omega f(x,u)w \, dx \tag{A.8}$$

for every $w \in W_0^{1,p}(\Omega)$ satisfying $w \geqslant 0$ a.e. in Ω. Given $0 \leqslant w \in W_0^{1,p}(\Omega)$, there exists a sequence $\{w_k\}_{k=1}^\infty \subset C_c^\infty(\Omega)$ such that $w_k \geqslant 0$ in Ω and $w_k \to w$ strongly in $W_0^{1,p}(\Omega)$ as $k \to \infty$. Since $p < q+1 \leqslant p^*$, this entails $w_k \to w$ strongly also in $L^{q+1}(\Omega)$ as $k \to \infty$. Moreover, we can find a subsequence, denoted again by $\{w_k\}_{k=1}^\infty$, such that $w_k \to w$ almost everywhere in Ω as $k \to \infty$. In Eq. (1.3) we now replace ϕ by w_k and apply Fatou's lemma to the integral $\int_\Omega g(u)w_k \, dx$ as $k \to \infty$, thus arriving at the desired inequality (A.8).

In particular, inequality (A.8) implies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w \, dx \geqslant \int_\Omega g(u)w \, dx \tag{A.9}$$

whenever $0 \leqslant w \in W_0^{1,p}(\Omega)$. Now we are ready to compare u with the weak solution $\underline{u}$ of problem (PS) given by Lemma A.4. We apply the weak comparison principle to (the weak formulation of) problem (PS) (with $\underline{u}$ in place of u) and to inequality (A.9) (with u), thus obtaining $u \geqslant \underline{u}$ a.e. in Ω. This guarantees $u \geqslant \epsilon_0\varphi_1$ a.e. in Ω.

Next, there are constants $0 < \ell < L < \infty$ such that $\ell d(x,\partial\Omega) \leqslant \varphi_1(x) \leqslant L d(x,\partial\Omega)$ for all $x \in \Omega$. It follows that $u \geqslant \epsilon_0 \ell d(\cdot,\partial\Omega)$ a.e. in Ω. Now, instead of using Fatou's lemma in the limiting process above, we apply Hardy's inequality to the integral $\int_\Omega g(u)w_k \, dx$ as $k \to \infty$, thus arriving at the desired equality (A.7) for every $w \in W_0^{1,p}(\Omega)$ satisfying $w \geqslant 0$ a.e. in Ω.

Finally, we make use of the polar decomposition $w = w^+ - w^-$ of an arbitrary function $w \in W_0^{1,p}(\Omega)$, where $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$ satisfy $w^+, w^- \in W_0^{1,p}(\Omega)$ and $\nabla w = \nabla w^+ - \nabla w^-$. Since we have already verified Eq. (A.7) for w^+ and w^-, the desired equality (A.7) holds also for every $w \in W_0^{1,p}(\Omega)$. □

Lemma A.6. *Let the assumptions* (f1)–(f2) *and* (g1)–(g2) *be satisfied. Each positive weak solution* u *of* (P) *belongs to* $L^\infty(\Omega)$ *and*

$$\|u\|_{L^\infty(\Omega)} \leqslant C_0$$

where C_0 *depends only on* $\|u\|_{W_0^{1,p}(\Omega)}$, N, p, q, Ω, C *(in* (f2)*) and* c_2.

Proof. First, we show that each positive weak solution u of (P) satisfies

$$\int_\Omega \left|\nabla(u-1)^+\right|^{p-2}\nabla(u-1)^+ \cdot \nabla w \, dx \leqslant \int_\Omega \big(g(1) + f(x,u)\big)w \, dx \tag{A.10}$$

for every $w \in C_c^\infty(\Omega)$ with $w \geqslant 0$. Indeed, let $\psi : \mathbb{R} \to [0,1]$ be a C^1 cut-off function such that $\psi(s) = 0$ if $s \leqslant 0$, $\psi'(s) \geqslant 0$ if $0 \leqslant s \leqslant 1$, and $\psi(s) = 1$ if $s \geqslant 1$. Given any $\epsilon > 0$, define $\psi_\epsilon(t) \stackrel{\text{def}}{=} \psi((t-1)/\epsilon)$ for $t \in \mathbb{R}$. Hence, $\psi_\epsilon \circ u \in W_0^{1,p}(\Omega)$ with $\nabla(\psi_\epsilon \circ u) = (\psi'_\epsilon \circ u)\nabla u$. Using the weak form of problem (P), Eq. (A.7), with the test function $\phi = (\psi_\epsilon \circ u)w$, where $w \in C_c^\infty(\Omega)$ satisfies $w \geqslant 0$, we obtain

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\big[(\psi_\epsilon \circ u)w\big] \, dx = \int_\Omega \big(g(u) + f(x,u)\big)(\psi_\epsilon \circ u)w \, dx.$$

Hence,

$$\int_\Omega |\nabla u|^p (\psi'_\epsilon \circ u)w \, dx + \int_\Omega |\nabla u|^{p-2}(\nabla u \cdot \nabla w)(\psi_\epsilon \circ u) \, dx = \int_\Omega \big(g(u) + f(x,u)\big)(\psi_\epsilon \circ u)w \, dx$$

with $\psi'_\epsilon \circ u \geqslant 0$, which yields

$$\int_\Omega |\nabla u|^{p-2}(\nabla u \cdot \nabla w)(\psi_\epsilon \circ u)\,\mathrm{d}x \leqslant \int_\Omega \big(g(u) + f(x,u)\big)(\psi_\epsilon \circ u) w\,\mathrm{d}x.$$

Letting $\epsilon \to 0+$ we arrive at (A.10). Finally, the L^∞ bound and regularity of u are obtained directly from Eq. (A.10) as follows: If $q < p^* - 1$, one applies Theorem A.1 from Anane [3], and if $q = p^* - 1$, the bootstrapping arguments from the proof of Theorem A.1, pp. 950–953, in García Azorero and Peral [5] to get a bound in $L^{\beta p^*}(\Omega)$ and Theorem 7.1 in Ladyženskaja and Ural'ceva [21] yield the desired result. In both Refs. [3,5] the bootstrapping arguments use the technique due to Serrin [24] (proof of Theorem 1). □

Finally, we are ready to bound any weak solution u of problem (P) by a positive scalar multiple of the eigenfunction φ_1 also from above. This result complements the corresponding bound from below, $u \geqslant \epsilon_0\varphi_1$ a.e. in Ω, stated in the first part of Lemma A.5 above. Equivalently, these lower and upper bounds for u/φ_1 can be reformulated as follows, using the distance function d in place of φ_1:

Lemma A.7. *Let the assumptions* (f1)–(f2) *and* (g1)–(g2) *be satisfied. Each positive weak solution u of problem* (P) *satisfies*

$$C_1 d(x,\partial\Omega) \leqslant u \leqslant K_1 d(x,\partial\Omega)$$

a.e. in Ω, where $0 < C_1 \leqslant K_1 < \infty$ are some constants independent of u, K_1 dependent of $\|u\|_{L^\infty(\Omega)}$.

Proof. Let $u \in W_0^{1,p}(\Omega)$ be a positive weak solution of problem (P). It follows from the first part of Lemma A.5 and its proof that $u(x) \geqslant \underline{u}(x) \geqslant \epsilon_0\varphi_1(x) \geqslant \epsilon_0 \ell d(x,\partial\Omega)$ for a.e. $x \in \Omega$. Hence, we can take $C_1 = \epsilon_0\ell > 0$ to get $u \geqslant C_1 d(\cdot,\partial\Omega)$ a.e. in Ω.

Next, we take advantage of the inequality $u \geqslant C_1 d(\cdot,\partial\Omega)$ to derive also $u \leqslant K_1 d(\cdot,\partial\Omega)$. Recall that $u \in L^\infty(\Omega)$, by Lemma A.6 above. First, we apply the estimate

$$g(u) + f(x,u) \leqslant C_1 \frac{1 + u^\delta + u^{q+\delta}}{u^\delta} \leqslant C \frac{1 + \|u\|^{q+\delta}_{L^\infty(\Omega)}}{u^\delta} \quad \text{a.e. in } \Omega$$

to the right-hand side of the equation in problem (P) to conclude that

$$\begin{cases} -\Delta_p u \leqslant C\big(1 + \|u\|^{q+\delta}_{L^\infty(\Omega)}\big)u^{-\delta} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0, & \text{in } \Omega \end{cases} \tag{A.11}$$

for some constant $C > 0$ large enough. After the substitution

$$v = \big(C + C\|u\|^{q+\delta}_{L^\infty(\Omega)}\big)^{-1/(p-1+\delta)} u,$$

inequality (A.11) is equivalent to

$$\begin{cases} -\Delta_p v \leqslant v^{-\delta} & \text{in } \Omega, \\ v|_{\partial\Omega} = 0, \quad v > 0, & \text{in } \Omega. \end{cases} \tag{A.12}$$

Let $\overline{u}$ the unique weak solution to

$$\text{(PSD)} \quad \begin{cases} -\Delta_p u = u^{-\delta} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0, & \text{in } \Omega. \end{cases}$$

in Ψ (see Lemma A.4). Now, in analogy with the proof of Lemma A.5, we apply the weak comparison principle to problem (PSD) (with $\overline{u}$ in place of u) and to inequality (A.12) (with v), thus arriving at $v \leqslant \overline{u}$ a.e. in Ω. Thus, it remains to verify $\overline{u} \leqslant c' d(\cdot,\partial\Omega)$ a.e. in Ω, where $0 < c' < \infty$ is a constant. This will imply $u \leqslant K_1 d(\cdot,\partial\Omega)$ a.e. in Ω with

$$K_1 = c'\big(C + C\|u\|^{q+\delta}_{L^\infty(\Omega)}\big)^{1/(p-1+\delta)}.$$

Thanks to $\ell d(x,\partial\Omega) \leqslant \varphi_1(x) \leqslant L d(x,\partial\Omega)$ for all $x \in \Omega$, with some constants $0 < \ell < L < \infty$, the inequality $\overline{u} \leqslant c' d(\cdot,\partial\Omega)$ in Ω is equivalent to $\overline{u} \leqslant c''\varphi_1$ in Ω, where $0 < c'' < \infty$ is a constant. We now construct a supersolution w to problem (PSD) of the form $w = \beta \cdot \Theta_\alpha \circ \varphi_1$ in Ω. Here, $\alpha, \beta > 0$ are suitable numbers and $\Theta_\alpha : [0, R_\alpha) \to \mathbb{R}_+$ is a C^1 function (where $0 < R_\alpha < \infty$ and $\mathbb{R}_+ = [0,\infty)$) that satisfies the initial value problem

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}r}\big(\big|\Theta'_\alpha(r)\big|^{p-2}\Theta'_\alpha(r)\big) = \Theta_\alpha(r)^{-\delta}, \quad 0 < r < R_\alpha, \\ \Theta_\alpha(0) = 0, \quad \Theta'_\alpha(0) = \alpha > 0. \end{cases} \tag{A.13}$$

The endpoint R_α is defined to be the supremum of all numbers $s \in (0,\infty)$ such that $\Theta'_\alpha(r) > 0$ holds for all $r \in [0,s)$. We will see that $0 < R_\alpha < \infty$ together with $\Theta'_\alpha(r) \searrow 0$ as $r \nearrow R_\alpha$.

Making use of the transformation

$$\begin{cases} \Theta_\alpha(r) = \alpha^{\frac{p}{1-\delta}} \cdot \Theta_1\big(\alpha^{-\frac{p}{p-1+\delta}} r\big), \quad 0 \leqslant r \leqslant R_\alpha, \\ R_\alpha = \alpha^{\frac{p}{p-1+\delta}} R_1, \end{cases} \tag{A.14}$$

we conclude that it suffices to treat the case $\alpha = 1$. Problem (A.13) with $\alpha = 1$ has the first integral

$$\begin{cases} -\dfrac{p-1}{p}\big|\Theta'_1(r)\big|^p - \dfrac{1}{1-\delta}\Theta_1(r)^{1-\delta} + C = 0, \quad 0 \leqslant r < R_1, \\ \Theta_1(0) = 0, \quad \Theta'_1(0) = 1 > 0, \end{cases} \tag{A.15}$$

where the constant C is given by $C = (p-1)/p$. There exists precisely one C^1 function $\Theta_1 : [0,R_1) \to \mathbb{R}_+$ that satisfies (A.15) together with $\Theta'_1(r) > 0$ for all $r \in [0,R_1)$; it is determined from

$$\int_0^{\Theta_1(r)} \left(1 - \frac{p}{(p-1)(1-\delta)}\,\theta^{1-\delta}\right)^{-1/p} \mathrm{d}\theta = r, \quad 0 \leqslant r < R_1, \tag{A.16}$$

where

$$\begin{aligned} R_1 &= \int_0^{[(p-1)(1-\delta)/p]^{1/(1-\delta)}} \left(1 - \frac{p}{(p-1)(1-\delta)}\theta^{1-\delta}\right)^{-1/p} \mathrm{d}\theta \\ &= \left(\frac{(p-1)(1-\delta)}{p}\right)^{1/(1-\delta)} \int_0^1 \big(1 - t^{1-\delta}\big)^{-1/p}\,\mathrm{d}t < \infty \end{aligned} \tag{A.17}$$

is the maximal number such that $\Theta'_1(r) > 0$ for all $r \in [0,R_1)$.

Let us first fix $\alpha > 0$ large enough, such that $R_\alpha > M \stackrel{\text{def}}{=} \max_{\overline{\Omega}} \varphi_1$. In the following calculations we make use of equations satisfied by ϕ_1 and (A.13) by Θ_α, respectively. The function $w(x) = \beta \cdot \Theta_\alpha(\varphi_1(x))$ of $x \in \Omega$ satisfies

$$\begin{aligned} &\nabla w(x) = \beta \cdot \Theta'_\alpha\big(\varphi_1(x)\big)\nabla\varphi_1(x), \\ &\big|\nabla w(x)\big|^{p-2}\nabla w(x) = \beta^{p-1}\big[\Theta'_\alpha\big(\varphi_1(x)\big)\big]^{p-1}\big|\nabla\varphi_1(x)\big|^{p-2}\nabla\varphi_1(x), \end{aligned}$$

whence

$$\begin{aligned} -\Delta_p w &= -\beta^{p-1}\big[\big(\big(\Theta'_\alpha\big)^{p-1}\big)' \circ \varphi_1\big]|\nabla\varphi_1|^p + \beta^{p-1}\big[\big(\big(\Theta'_\alpha\big)^{p-1}\big) \circ \varphi_1\big](-\Delta_p\varphi_1) \\ &= \beta^{p-1}(\Theta_\alpha \circ \varphi_1)^{-\delta}|\nabla\varphi_1|^p + \beta^{p-1}\lambda_1\big[\big(\big(\Theta'_\alpha\big)^{p-1}\big) \circ \varphi_1\big] \cdot \varphi_1^{p-1} \\ &= \beta^{p-1+\delta}|\nabla\varphi_1|^p w^{-\delta} + \beta^{p-1}\lambda_1\big[\big(\big(\Theta'_\alpha\big)^{p-1}\big) \circ \varphi_1\big] \cdot \varphi_1^{p-1}(\beta \cdot \Theta_\alpha \circ \varphi_1)^\delta w^{-\delta} \\ &= \beta^{p-1+\delta}\big\{|\nabla\varphi_1|^p + \lambda_1\big[\big(\big(\Theta'_\alpha\big)^{p-1}\big) \circ \varphi_1\big] \cdot \varphi_1^{p-1}(\Theta_\alpha \circ \varphi_1)^\delta\big\} w^{-\delta}. \end{aligned} \tag{A.18}$$

Recall $R_\alpha > M = \max_{\overline{\Omega}} \varphi_1$. The function Θ_α being strictly increasing with strictly decreasing derivative Θ'_α on the interval $[0,R_\alpha]$, and $\Theta_\alpha(0) = 0$, $\Theta'_\alpha(0) = \alpha > \Theta'_\alpha(R_\alpha) = 0$, we can estimate

$$\begin{aligned} &\big(\big(\Theta'_\alpha\big)^{p-1}\big) \circ \varphi_1 \geqslant \Theta'_\alpha(M)^{p-1} > 0, \\ &\Theta_\alpha \circ \varphi_1 \geqslant \Theta'_\alpha(M)\varphi_1. \end{aligned}$$

We combine these inequalities to estimate the second summand in the curly brackets at the end of Eq. (A.18) above, thus obtaining

$$-\Delta_p w \geqslant \beta^{p-1+\delta}\big\{|\nabla\varphi_1|^p + \lambda_1\big(\Theta'_\alpha(M)\varphi_1\big)^{p-1+\delta}\big\} w^{-\delta}. \tag{A.19}$$

Moreover, we have $w \in C^1(\overline{\Omega})$ together with $w = 0$ on $\partial\Omega$, $w > 0$ in Ω, and $\frac{\partial w}{\partial \nu} < 0$ on $\partial\Omega$. These claims follow from $\varphi_1 \in C^1(\overline{\Omega})$ combined with the strong maximum and boundary point principles $\varphi_1 > 0$ in Ω and $\frac{\partial \varphi_1}{\partial \nu} < 0$ on $\partial\Omega$ (see Vázquez [27, Theorem 5, p. 200]). The same arguments render

$$\gamma \stackrel{\text{def}}{=} \min_{\overline{\Omega}}\big\{|\nabla\varphi_1|^p + \lambda_1\big(\Theta'_\alpha(M)\varphi_1\big)^{p-1+\delta}\big\} > 0.$$

We choose the number $\beta > 0$ large enough, such that $\beta^{p-1+\delta}\gamma \geqslant 1$. In particular, inequality (A.19) yields

$$-\Delta_p w \geqslant w^{-\delta} \quad \text{in } \Omega. \tag{A.20}$$

Finally, we apply the weak comparison principle to problem (PSD) (with $\bar{u}$ in place of u) and to inequality (A.20) (with w satisfying $w = 0$ on $\partial\Omega$), thus arriving at $w \geqslant \bar{u}$ a.e. in Ω. We have thus verified

$$v \leqslant \bar{u} \leqslant w = \beta \cdot \Theta_\alpha \circ \varphi_1 \leqslant \alpha\beta\varphi_1 \leqslant c' d(\cdot, \partial\Omega) \quad \text{a.e. in } \Omega,$$

where $c' \in (0, \infty)$ is a constant, as desired.

The proof of Lemma A.7 is now complete. □

Appendix B

In this appendix, we recall some results proved in [17]. Precisely, we consider the following quasilinear elliptic boundary value problem,

$$-\nabla \cdot \big(\mathbf{a}(x, \nabla u)\big) = f(x) \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega, \tag{B.1}$$

in a setting that is closely related to Lieberman's in [23, Theorem 1, p. 1203]. We assume that Ω is a (nonempty) bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a compact C^2 manifold. We denote by $x = (x_1, \dots, x_N)$ a generic point in Ω and by u the unknown function of x, where $u \in W_0^{1,p}(\Omega)$ for $p \in (1, \infty)$. The quasilinear elliptic operator $(x, u) \mapsto \nabla \cdot (\mathbf{a}(x, \nabla u))$ is defined by

$$\nabla \cdot \big(\mathbf{a}(x, \nabla u)\big) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\big(x, \nabla u(x)\big) \quad \text{for } x \in \Omega \text{ and } u \in W_0^{1,p}(\Omega) \tag{B.2}$$

with values in $W^{-1,p'}(\Omega)$, the dual space of $W_0^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. The components a_i of the vector field $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathbf{a} = (a_1, \dots, a_N)$, are functions of x and $\eta = \nabla u \in \mathbb{R}^N$, such that $a_i \in C^0(\Omega \times \mathbb{R}^N)$ and $\partial a_i/\partial\eta_j \in C^0(\Omega \times (\mathbb{R}^N \setminus \{0\}))$. We assume that $\mathbf{a}$ satisfies the following *ellipticity* and *growth conditions*:

(H1) There exist some constants $\kappa \in [0, 1]$, $\gamma, \Gamma \in (0, \infty)$, and $\alpha \in (0, 1)$, such that

$$a_i(x, 0) = 0, \quad i = 1, \dots, N, \tag{B.3}$$

$$\sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \eta_j}(x, \eta) \cdot \xi_i \xi_j \geqslant \gamma \cdot \big(\kappa + |\eta|\big)^{p-2} \cdot |\xi|^2, \tag{B.4}$$

$$\sum_{i,j=1}^{N} \left| \frac{\partial a_i}{\partial \eta_j}(x, \eta) \right| \leqslant \Gamma \cdot \big(\kappa + |\eta|\big)^{p-2}, \tag{B.5}$$

$$\sum_{i=1}^{N} \big|a_i(x, \eta) - a_i(y, \eta)\big| \leqslant \Gamma \cdot \big(1 + |\eta|\big)^{p} \cdot |x - y|^{\alpha}, \tag{B.6}$$

for all $x, y \in \Omega$, all $\eta \in \mathbb{R}^N \setminus \{0\}$, and all $\xi \in \mathbb{R}^N$.

We remark that conditions (B.3) through (B.6) are motivated by the elliptic boundary value problem

$$-\Delta_p u = f(x) \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega, \tag{B.7}$$

with the p-Laplacian defined by $\Delta_p u \stackrel{\text{def}}{=} \nabla \cdot (|\nabla u|^{p-2} \nabla u)$.

Finally, we impose the following *growth condition* on the function $f \in L^\infty_{\text{loc}}(\Omega)$:

(H2) There exist constants c and δ, $0 < c < \infty$ and $0 < \delta < 1$, such that

$$0 \leqslant f(x) \leqslant c d(x, \partial\Omega)^{-\delta} \quad \text{holds for almost all } x \in \Omega. \tag{B.8}$$

Then, we have the following analogue of a well-known regularity result for problem (B.1) due to Lieberman [23, Theorem 1, p. 1203] (regularity near the boundary). Interior regularity was established earlier independently by DiBenedetto [12, Theorem 2, p. 829] and Tolksdorf [26, Theorem 1, p. 127]. Theorem B.1 is proved in [17].

Theorem B.1. *Assume that* $\mathbf{a}(x,\eta)$ *satisfies the structural hypotheses* (B.3)–(B.6)*, and* $f(x)$ *satisfies the growth hypothesis* (B.8)*. Let* $u \in W_0^{1,p}(\Omega)$ *be the* (*unique*) *weak solution of problem* (B.1)*. In addition, assume*

$$0 \leqslant u(x) \leqslant Cd(x,\partial\Omega) \quad \textit{for almost all } x \in \Omega, \tag{B.9}$$

where C *is a constant,* $0 \leqslant C < \infty$*. Then there exist constants* β *and* M*,* $0 < \beta < \alpha$ *and* $0 \leqslant M < \infty$*, depending solely on* Ω*,* N*,* p*, on the constants* γ*,* Γ*,* α *in* (B.4) *through* (B.6)*, on the constants* c*,* δ *in* (B.8)*, and on the constant* C *in* (B.9)*, but* not *on* $\kappa \in [0,1]$*, such that* u *satisfies* $u \in C^{1,\beta}(\overline{\Omega})$ *and*

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \leqslant M. \tag{B.10}$$

Appendix C. (Summary)

Theorem 1.1 shows that for a class of quasilinear singular elliptic equations with boundary Dirichlet conditions, any $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ local minimizer for the associated energy functional is a $W_0^{1,p}(\Omega)$ minimizer. This result generalises Proposition 3.7 in [17] which only concerns the case $g(t) = t^{-\delta}$ and $f(x,t) = t^q$ with $0 < \delta < 1$ and $p - 1 < q < p^* - 1$. The proof used to prove Theorem 1.1 does not modify the quasilinear operator $-\Delta_p$ in (P) and then the $C^{1,\alpha}(\overline{\Omega})$-regularity is easier to get. Therefore, this approach could be considered for more general quasilinear operators in the form given in Appendix B to get multiplicity results for corresponding singular equations. It will be interesting to get similar results for anisotropic operators as the $p(x)$-Laplacian operator.

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